Summation of the progressions $\sin^\lambda \varphi + \sin^\lambda 2\varphi + \sin^\lambda 3\varphi + \cdots + \sin^\lambda n\varphi$ and $\cos^\lambda \varphi + \cos^\lambda 2\varphi + \cos^\lambda 3\varphi + \cdots + \cos^\lambda n\varphi$ *

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§1 Setting

$$\cos\varphi + \sqrt{-1}\cdot\sin\varphi = p$$

and

$$\cos\varphi - \sqrt{-1}\cdot\sin\varphi = q$$

it is known that

$$\cos n\varphi = \frac{p^n + q^n}{2}$$

and

$$\sin n\varphi = \frac{p^n - q^n}{2\sqrt{-1}},$$

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furthermore that $pq = 1$. Having constituted all this, it is evident that the summation of these series can always be reduced to the following two series, more precisely geometric progressions

$$p^{\alpha} + p^{2\alpha} + p^{3\alpha} + \cdots + p^{n\alpha} = \frac{p^{(n+1)\alpha} - p^{\alpha}}{p^{\alpha} - 1} = \frac{p^{\alpha}(1 - p^{n\alpha})}{1 - p^{\alpha}},$$

$$q^{\alpha} + q^{2\alpha} + q^{3\alpha} + \cdots + q^{n\alpha} = \frac{q^{(n+1)\alpha} - q^{\alpha}}{q^{\alpha} - 1} = \frac{q^{\alpha}(1 - q^{n\alpha})}{1 - q^{\alpha}}.$$

§2 If these two progressions are added such that this one results

$$p^{\alpha} + p^{2\alpha} + p^{3\alpha} + \cdots + p^{n\alpha} + q^{\alpha} + q^{2\alpha} + q^{3\alpha} + \cdots + q^{n\alpha},$$

its sum will be

$$\frac{p^{\alpha} - p^{(n+1)\alpha}}{1 - p^{\alpha}} + \frac{q^{\alpha} - q^{(n+1)\alpha}}{1 - q^{\alpha}}$$

$$\frac{p^{\alpha} - p^{(n+1)\alpha} - p^{\alpha}q^{\alpha} + p^{(n+1)\alpha}q^{\alpha} + q^{\alpha} - q^{(n+1)\alpha} - p^{\alpha}q^{\alpha} + p^{\alpha}q^{(n+1)\alpha}}{1 - p^{\alpha} - q^{\alpha} + p^{\alpha}q^{\alpha}},$$

which expression, because of $pq = 1$, is transformed into this one

$$\frac{p^{\alpha} - p^{(n+1)\alpha} - 1 + p^{n\alpha} + q^{\alpha} - q^{(n+1)\alpha} - 1 + q^{n\alpha}}{2 - p^{\alpha} - q^{\alpha}},$$

which further, because of

$$p^{\alpha} + q^{\alpha} = 2\cos\alpha\varphi$$

and

$$p^{(n+1)\alpha} + q^{(n+1)\alpha} = 2\cos(n+1)\alpha\varphi$$

and

$$p^{n\alpha} + q^{n\alpha} = 2\cos n\alpha\varphi,$$

is reduced to this form

$$\frac{\cos\alpha\varphi - \cos(n+1)\alpha\varphi - 1 + \cos n\alpha\varphi}{1 - \cos\alpha\varphi} = -1 + \frac{\cos n\alpha\varphi - \cos(n+1)\alpha\varphi}{1 - \cos\alpha\varphi},$$

which is thus the sum of the propounded series.

§**3** If the the second of our progressions is subtracted from the first such that one has this one

$$p^{\alpha}+p^{2\alpha}+p^{3\alpha}+\cdots+p^{n\alpha}-q^{\alpha}-q^{2\alpha}-q^{3\alpha}-\cdots-q^{n\alpha},$$

its sum will be

$$\frac{p^{\alpha}-p^{(n+1)\alpha}}{1-p^{\alpha}}-\frac{q^{\alpha}-q^{(n+1)\alpha}}{1-q^{\alpha}},$$

which parts brought to the common denominator will give

$$\begin{Bmatrix} +\,p^{\alpha} & -\,p^{(n+1)\alpha} & -\,p^{\alpha}q^{\alpha} & +\,p^{(n+1)\alpha}q^{\alpha} \\ -\,q^{\alpha} & +\,q^{(n+1)\alpha} & +\,p^{\alpha}q^{\alpha} & -\,q^{(n+1)\alpha}p^{\alpha} \end{Bmatrix} : (1-p^{\alpha}-q^{\alpha}+p^{\alpha}q^{\alpha});$$

since $pq=1$, this expression is reduced to

$$\begin{Bmatrix} p^{\alpha} & -\,p^{(n+1)\alpha} & -\,1+p^{n\alpha} \\ -\,q^{\alpha} & +\,q^{(n+1)\alpha} & +\,1-q^{n\alpha} \end{Bmatrix} : (2-p^{\alpha}-q^{\alpha})$$

and further because of

$$p^{\alpha}-q^{\alpha}=2\sqrt{-1}\cdot\sin\alpha\varphi$$

and

$$p^{(n+1)\alpha}-q^{(n+1)\alpha}=2\sqrt{-1}\cdot\sin(n+1)\alpha\varphi$$

and

$$p^{n\alpha}-q^{n\alpha}=2\sqrt{-1}\cdot\sin n\alpha\varphi,$$

it is transformed into this expression

$$\frac{\sin\alpha\varphi-\sin(n+1)\alpha\varphi+\sin n\alpha\varphi}{1-\cos\alpha\varphi}\sqrt{-1}.$$

§**4** For the sake of brevity, let us denote the sums of these series by writing the summation sign $\int$ in front of the last term such that the two cases that we discussed give the following summations

$$\int (p^{n\alpha}+q^{n\alpha}) = -1+\frac{\cos n\alpha\varphi-\cos(n+1)\alpha\varphi}{1-\cos\alpha\varphi}$$

and

$$\int (p^{n\alpha}-q^{n\alpha}) = \frac{\sin\alpha\varphi+\sin n\alpha\varphi-\sin(n+1)\alpha\varphi}{1-\cos\alpha\varphi}\sqrt{-1};$$

it will be easy to deduce all cases that are propounded from these formulas.

§**5** Therefore, first let $\lambda=1$ such that one has to sum these two series

$$s=\sin\varphi+\sin 2\varphi+\sin 3\varphi+\cdots+\sin n\varphi$$

or

$$s=\int \sin n\varphi$$

and

$$t=\cos\varphi+\cos 2\varphi+\cos 3\varphi+\cdots+\cos n\varphi$$

or

$$t=\int \cos n\varphi;$$

since

$$\sin n\varphi=\frac{p^n-q^n}{2\sqrt{-1}}$$

and

$$\cos n\varphi=\frac{p^q+q^n}{2},$$

we will have

$$2s\sqrt{-1}=\int (p^n-q^n)$$

and

$$2t = \int (p^n + q^n)$$

such that from the preceding paragraph, because of $\alpha = 1$, we obtain immediately

$$2s\sqrt{-1} = \frac{\sin\varphi + \sin n\varphi - \sin(n+1)\varphi}{1 - \cos\varphi}\sqrt{-1}$$

and

$$2t = -1 + \frac{\cos n\varphi - \cos(n+1)\varphi}{1 - \cos\varphi}$$

and hence

$$s = \frac{\sin\varphi + \sin n\varphi - \sin(n+1)\varphi}{2(1 - \cos\varphi)}$$

and

$$t = -\frac{1}{2} + \frac{\cos n\varphi - \cos(n+1)\varphi}{2(1 - \cos\varphi)}.$$

§6 Now let $\lambda = 2$, and again set

$$s = \sin^2\varphi + \sin^2 2\varphi + \cdots + \sin^2 n\varphi$$

or

$$s = \int \sin^2 n\varphi$$

and

$$t = \cos^2\varphi + \cos^2 2\varphi + \cdots + \cos^2 n\varphi$$

or

$$t = \int \cos^2 n\varphi;$$

since

$$\sin^2 n\varphi = \frac{p^{2n} - 2p^n q^n + q^{2n}}{-4} = \frac{1}{2} - \frac{p^{2n} + q^{2n}}{4}$$

and

$$\cos^2 n\varphi = \frac{p^{2n} + 2p^n q^n + q^{2n}}{4} = \frac{1}{2} + \frac{p^{2n} + q^{2n}}{4},$$

we will have these formulas

$$4s = 2\int 1 - \int (p^{2n} + q^{2n})$$

and

$$4t = 2\int 1 + \int (p^{2n} + q^{2n}),$$

where, since the number of terms is n, obviously $\int 1 = n$; since $\alpha = 2$, from the above results

$$\int (p^{2n} + q^{2n}) = -1 + \frac{\cos 2n\varphi - \cos 2(n+1)\varphi}{1 - \cos 2\varphi},$$

after the substitution of these values and a division by 4 we will obtain

$$s = \frac{n}{2} + \frac{1}{4} - \frac{\cos 2n\varphi - \cos 2(n+1)\varphi}{4(1 - \cos 2\varphi)}$$

and

$$t = \frac{n}{2} - \frac{1}{4} + \frac{\cos 2n\varphi - \cos 2(n+1)\varphi}{4(1 - \cos 2\varphi)}$$

and hence it is clear immediately that

$$s + t = n,$$

as the matter of things require it.

§7 Let us set $\lambda = 3$ now and represent the series that we have to sum in this way

$$s = \sin^3 \varphi + \sin^3 2\varphi + \cdots + \sin^3 n\varphi$$

or

$$s = \int \sin^3 n\varphi$$

and

$$t = \cos^3 \varphi + \cos^3 2\varphi + \cdots + \cos^3 n\varphi$$

or

$$t = \int \cos^3 n\varphi.$$

Since

$$\sin^3 n\varphi = \frac{p^{3n} - 3p^{2n}q^n + 3p^n q^{2n} - q^{3n}}{-8\sqrt{-1}}$$

and

$$\cos^3 n\varphi = \frac{p^{3n} + 3p^{2n}q^n + 3p^n q^{2n} + q^{3n}}{8},$$

and because of $pq = 1$ we obtain

$$s = \frac{-1}{8\sqrt{-1}} \int \left(p^{3n} - q^{3n} - 3p^n + 3q^n\right) = \frac{-1}{8\sqrt{-1}} \int \left(p^{3n} - q^{3n}\right) + \frac{3}{8\sqrt{-1}} \int \left(p^n - q^n\right),$$

but then

$$t = +\frac{1}{8} \int \left(p^{3n} + q^{3n}\right) + \frac{3}{8} \int \left(p^n + q^n\right);$$

if we substitute the values that we found above here, both sums in question will result expressed in this way

$$s = \frac{-\sin 3\varphi - \sin 3n\varphi + \sin 3(n+1)\varphi}{8(1-\cos 3\varphi)} + \frac{3\sin\varphi + 3\sin n\varphi - 3\sin(n+1)\varphi}{8(1-\cos\varphi)},$$

$$t = -\frac{1}{2} + \frac{\cos 3n\varphi - \cos 3(n+1)\varphi}{8(1-\cos 3\varphi)} + \frac{3n\cos\varphi - 3\cos(n+1)\varphi}{8(1-\cos\varphi)}.$$

§8 Let $\lambda = 4$ such that these sums are in question

$$s = \sin^4\varphi + \sin^4 2\varphi + \cdots + \sin^4 n\varphi$$

or

$$s = \int \sin^4 n\varphi$$

and

$$t = \cos^4\varphi + \cos^4 2\varphi + \cdots + \cos^4 n\varphi$$

or

$$t = \int \cos^4 n\varphi.$$

Since

$$\sin^4 n\varphi = \frac{p^{4n} - 4p^{3n}q^n + 6p^{2n}q^{2n} - 4p^n q^{3n} + q^{4n}}{16}$$

and

$$\cos^4 n\varphi = \frac{p^{4n} + 4p^{3n}q^n + 6p^{2n}q^{2n} + 4p^n q^{3n} + q^{4n}}{16}$$

and because of $pq = 1$ these values follow

$$s = \frac{1}{16}\int\left(p^{4n} + q^{4n}\right) - \frac{1}{4}\int\left(p^{2n} + q^{2n}\right) + \frac{3}{8}\int 1$$

and

$$t = \frac{1}{16}\int\left(p^{4n} + q^{4n}\right) + \frac{1}{4}\int\left(p^{2n} + q^{2n}\right) + \frac{3}{8}\int 1;$$

having substituted the values that we gave above we will find

$$s = \frac{3n}{8} + \frac{3}{16} + \frac{\cos 4n\varphi - \cos 4(n+1)\varphi}{16(1-\cos 4\varphi)} - \frac{\cos 2n\varphi - \cos 2(n+1)\varphi}{4(1-\cos 2\varphi)}$$

and

$$t = \frac{3n}{8} - \frac{5}{16} + \frac{\cos 4n\varphi - \cos 4(n+1)\varphi}{16(1-\cos 4\varphi)} + \frac{\cos 2n\varphi - \cos 2(n+1)\varphi}{4(1-\cos 2\varphi)}$$

and so it will be easy to expand even greater values of the exponent λ.

§9 If it is question sums of which kind will result, if those series are continued to infinity, one will have to be very careful. For, if the exponent λ was an even number, it is evident that for an infinite number n the sums of these series will also be infinitely large; on the other hand, if λ is an odd number, there is no reason why these sums should become infinite; for, then the whole question reduces to this that the values of the formulas $\sin n\alpha\varphi$ and $\cos n\alpha\varphi$ are assigned, if an infinitely large number is taken for n; but it is perspicuous that these values can vary from the limit -1 to the limit $+1$ in this case, as if n would be a finite number; if this is considered separately, nothing certain can be confirmed about these sums, since, whatever sum is presented, if additionally one or more terms are added, a completely different sum would result. Nevertheless, the illustrious author of the preceding dissertation assigned the sums in this case using arguments from metaphysics in an ingenious way such that we can be satisfied in analysis.

§10 Since in these series as in all other non convergent series the notion of a *sum* does not make any sense, since, no matter how many terms are actually added, one nevertheless never arrives at a definite sum, I, basing it on most solid reasons, emphasised that in these cases another meaning has to be attributed to the word sum; I think that this new notion must be constituted in such a way that a sum of an infinite series, no matter whether it converges or diverges, is the analytical formula from the expansion of which the series originated; and having admitted this definition, all doubts about summations of this kind vanish immediately.

§11 That it becomes more clear, let us consider the first series exhibited above, i.e.

$$s = \sin\varphi + \sin 2\varphi + \sin 3\varphi + \cdots + \sin n\varphi,$$

for which we found

$$\frac{\sin\varphi + \sin n\varphi - \sin(n+1)\varphi}{2(1-\cos\varphi)};$$

the formulas $\sin n\varphi$ and $\sin(n+1)\varphi$ enter into this expression because of the last term; therefore, if this series would actually be continued to infinity these formulas would go out of the sum such that in this case

$$s = \frac{\sin \varphi}{2(1 - \cos \varphi)},$$

which is the formula, from the expansion of which this series is found; hence according to my definition, this formula can be considered as the sum of this series; the same is true for the other series

$$t = \cos \varphi + \cos 2\varphi + \cos 3\varphi + \cdots + \cos n\varphi,$$

for which we found

$$t = -\frac{1}{2} + \frac{\cos n\varphi - \cos(n+1)\varphi}{2(1 - \cos \varphi)};$$

for, omitting the last member that depends just on the last term, by my definition we will find $t = -\frac{1}{2}$; since this is not seen that easily, one has to note that this value originated from the formula

$$t = \frac{\cos \varphi - 1}{2(1 - \cos \varphi)};$$

that this value is equal to the propounded series can be shown in this way: Multiply both sides by $2 - 2\cos \varphi$ and it must be

$$\cos \varphi - 1 = 2\cos \varphi + 2\cos 2\varphi + 2\cos 3\varphi + 2\cos 4\varphi + \text{etc.}$$
$$-2\cos^2 \varphi - 2\cos \varphi \cos 2\varphi - 2\cos \varphi \cos 3\varphi - \text{etc.};$$

since in general

$$2\cos a \cos b = \cos(a - b) + \cos(a + b),$$

it will be

$$2\cos^2\varphi = 1 + \cos 2\varphi,$$
$$2\cos\varphi\cos 2\varphi = \cos\varphi + \cos 3\varphi,$$
$$2\cos\varphi\cos 3\varphi = \cos 2\varphi + \cos 4\varphi,$$
$$2\cos\varphi\cos 4\varphi = \cos 3\varphi + \cos 5\varphi,$$
$$2\cos\varphi\cos 5\varphi = \cos 4\varphi + \cos 6\varphi,$$
$$2\cos\varphi\cos 6\varphi = \cos 5\varphi + \cos 7\varphi$$
etc.;

having substituted these values the equality is obvious, of course; for, it will result

$$\begin{aligned}\cos\varphi - 1 = 2\cos\varphi &+ 2\cos 2\varphi + 2\cos 3\varphi + 2\cos 2\varphi + \text{etc.}\\ -1 - 2\cos\varphi &- \cos 2\varphi - \cos 3\varphi - \cos 4\varphi - \text{etc.}\\ &- \cos 2\varphi - \cos 3\varphi - \cos 4\varphi - \text{etc.}\end{aligned}$$

§**12** Having observed these things, even for the case $\lambda = 3$, for which we set

$$s = \sin^3\varphi + \sin^3 2\varphi + \sin^3 3\varphi + \sin^4\varphi + \text{etc. to infinity}$$

and

$$t = \cos^3\varphi + \cos^3 2\varphi + \cos^3 3\varphi + \cos^3 4\varphi + \text{etc. to infinity},$$

the sums of these infinite series will be expressed in this way

$$s = -\frac{\sin 3\varphi}{8(1-\cos 3\varphi)} + \frac{3\sin\varphi}{8(1-\cos\varphi)} \quad \text{and} \quad t = -\frac{1}{2};$$

it is not immediately clear that the expansions of these formulas lead to these series; nevertheless, it is certain that this is a perfect equality, which will be understood by those who are experienced in this kind of calculus. Despite all this, it will be illustrative to show the validity of the last summation in this way: Since

$$\cos^3 a = \frac{3}{4}\cos a + \frac{1}{4}\cos 3a,$$

this series is resolved into the two following ones

$$t = \frac{3}{4}(\cos\varphi \quad + \cos 2\varphi + \cos 4\varphi + \text{etc.})$$

$$+ \frac{1}{4}(\cos 3\varphi + \cos 6\varphi + \cos 9\varphi + \text{etc.}),$$

but from the preceding the sum of the first series is

$$\frac{3}{4} \cdot -\frac{1}{2} = -\frac{3}{8},$$

the sum of the second sum, for the same reason, reads as

$$\frac{1}{4} \cdot -\frac{1}{2} = -\frac{1}{8},$$

whence both combined give the sum

$$-\frac{1}{2}.$$

General Summation of other infinite Progressions of this kind

Theorem

If the sum of this progression was known

$$Az + Bz^2 + Cz^3 + Dz^4 + \cdots + Nz^n,$$

it will always be possible to sum the following progressions, too

$$S = Ax\sin\varphi + Bx^2\sin 2\varphi + Cx^3\sin 3\varphi + \cdots + Nx^n\sin n\varphi$$

and

$$T = Ax\cos\varphi + Bx^2\cos 2\varphi + Cx^3\cos 3\varphi + \cdots + Nx^n\cos n\varphi.$$

Proof

Since the sum of the progression

$$Az + Bz^2 + Cz^3 + \cdots + Nz^n$$

is a certain function of the variable quantity z, denote it by the formula $\Delta : z$; then, as before, setting

$$p = \cos\varphi + \sqrt{-1} \cdot \sin\varphi$$

and

$$q = \cos\varphi - \sqrt{-1} \cdot \sin\varphi$$

such that

$$\sin n\varphi = \frac{1}{2\sqrt{-1}}(p^n - q^n)$$

and

$$\cos n\varphi = \frac{p^n + q^n}{2},$$

if these formulas are substituted in the propounded series, their sums will be obtained expressed in this way:

$$2S\sqrt{-1} = \Delta : px - \Delta : qx$$

and

$$2T = \Delta : px + \Delta : qx,$$

where one has to note that in each of both formulas the imaginary quantities connected to the letters p and q cancel each other such that real values will result for the sums S and T; and this summation will succeed, no matter whether the propounded series is continued to infinity or terminates at some point.

EXAMPLE 1

Let all coefficients $A, B, C, \cdots$ be $= 1$ and continue the series to infinity; thus,

$$\Delta : z = \frac{z}{1-z};$$

therefore, for the first series

$$S = x\sin\varphi + x^2\sin 2\varphi + x^3\sin 3\varphi + x^4\sin 4\varphi + \cdots \text{ to infinity}$$

we will have

$$2S\sqrt{-1} = \frac{px}{1-px} - \frac{qx}{1-qx} = \frac{(p-q)x}{1-(p+q)x+pqx^2},$$

which equation because of

$$p - q = 2\sqrt{-1}\cdot\sin\varphi$$

and

$$p + q = 2\cos\varphi$$

and $pq = 1$ will give

$$S = \frac{x\sin\varphi}{1-2x\cos\varphi+x^2}.$$

For the other series

$$T = x\cos\varphi + x^2\cos 2\varphi + x^3\cos 3\varphi + x^4\cos 4\varphi + \cdots \text{ to infinity}$$

we will have

$$2T = \frac{px}{1-px} + \frac{qx}{1-qx} = \frac{(p+q)x-2pqx^2}{1-(p+q)x+pqx^2}$$

or

$$T = \frac{x\cos\varphi - x^2}{1-2x\cos\varphi+x^2}.$$

COROLLARY 1

Therefore, if $x = 1$, the summations given above result, i.e.

$$S = \frac{\sin\varphi}{2(1-\cos\varphi)} = \frac{1}{2}\cot\frac{1}{2}\varphi$$

and

$$T = -\frac{1}{2},$$

which case is even more remarkable, since each term is a variable quantity, although the sum is a constant quantity.

COROLLARY 2

But often it will be possible to assume the quantity x in such a way that the sum of the given series becomes equal to a quantity a; for the first series of sines we will have

$$\frac{x\sin\varphi}{1-2x\cos\varphi+x^2} = a;$$

and if the letter a is determined from this equation, it will certainly be

$$a = x\sin + x^2\sin\sin 2\varphi + x^3\sin 3\varphi + \cdots,$$

and in like manner, if we set

$$\frac{x\cos\varphi - x^2}{1-2x\cos\varphi+x^2} = a,$$

find the letter x from this; we will thus also have

$$a = x\cos\varphi + x^2\cos 2\varphi + x^3\cos 3\varphi + \cdots$$

EXAMPLE 2

Now let

$$\Delta : z = z + \frac{1}{2}z^2 + \frac{1}{3}z^3 + \frac{1}{4}z^4 + \cdots \text{to infinity} = \log\frac{1}{1-z}$$

such that the propounded series are

$$S = x \sin \varphi + \frac{1}{2} x^2 \sin 2\varphi + \frac{1}{3} x^3 \sin 3\varphi + \cdots$$

and

$$T = x \cos \varphi + \frac{1}{2} x^2 \cos 2\varphi + \frac{1}{3} x^3 \cos 3\varphi + \cdots ;$$

we will have

$$2S\sqrt{-1} = \log \frac{1}{1-px} - \log \frac{1}{1-qx} = \log \frac{1-qx}{1-px}$$

or

$$2S\sqrt{-1} = \log \frac{1 - x\cos\varphi + x\sqrt{-1} \cdot \sin\varphi}{1 - x\cos\varphi - x\sqrt{-1} \cdot \sin\varphi};$$

for the reduction of this formula consider this form

$$\log \frac{f + g\sqrt{-1}}{f - g\sqrt{-1}},$$

about which we know, if we set

$$\frac{g}{f} = \tan \omega,$$

that this logarithm will be $= 2\omega\sqrt{-1}$; therefore, find the angle ω such that

$$\tan \omega = \frac{x \sin \varphi}{1 - x\cos\varphi},$$

whence it follows

$$S = \omega.$$

For the other progression, since

$$2T = \log \frac{1}{1-px} + \log \frac{1}{1-qx} = -\log(1 - 2x\cos\varphi + x^2),$$

we have

$$T = -\frac{1}{2} \log(1 - 2x\cos\varphi + x^2).$$

COROLLARY

Since for the first progression

$$\frac{x \sin \varphi}{1 - x \cos \varphi} = \tan \omega,$$

from this we find

$$x = \frac{\tan \omega}{\sin \varphi + \cos \varphi \tan \omega} = \frac{\sin \omega}{\sin(\varphi + \omega)};$$

having substituted this value we will obtain this remarkable summation

$$\omega = \frac{\sin \omega \sin \varphi}{\sin(\varphi + \omega)} + \frac{\sin^2 \omega \sin 2\varphi}{2 \sin^2(\varphi + \omega)} + \frac{\sin^3 \omega \sin 3\varphi}{3 \sin^3(\varphi + \omega)} + \frac{\sin^4 \omega \sin 4\varphi}{4 \sin^4(\varphi + \omega)} + \cdots;$$

if $\omega = \frac{\pi}{2}$ such that $\sin \omega = 1$ and $\sin(\varphi + \omega) = \cos \varphi$, this extraordinary summation results

$$\frac{\pi}{2} = \frac{\sin \varphi}{1 \cos \varphi} + \frac{\sin 2\varphi}{2 \cos^2 \varphi} + \frac{\sin 3\varphi}{3 \cos^3 \varphi} + \cdots,$$

which series I already obtain in my book *Institutiones calculi differentialis* from most different principles; it seemed to be even more remarkable, since, no matter what is taken for φ, the sum always remains the same, i.e. $= \frac{\pi}{2}$.